ABSTRACT
Dynamic characteristics of active magnetic bearing (AMB)-flexible rotor system are closely related to control law. To analyze dynamic characteristics of flexible rotor suspended by AMBs with linear quadratic regulation (LQR) controller, a simple and effective method based on numerical calculation of unbalanced response is proposed in this article. The model of flexible rotor is established based upon Euler-Bernoulli beam theory and Lagrange’s equation. Disc on the rotor and its Gyro effect are taken into account. LQR controller based on error and its derivative is developed to control electromagnetic force of AMB at each degree of freedom (DOF) in real time. Under the unbalanced exciting force, the steady-state response and transient response in time domain of each node of flexible rotor at 0-4000 rad/s are calculated numerically. The critical speeds of rotor are obtained by identification method quickly and easily.

INTRODUCTION
In order to improve capacity of providing high-temperature heat, the High Temperature gas-cooled Reactor, constructed in Shandong Province, China, is coupled with helium-turbine circle instead of steam-turbine [1-4]. Helium Turbine Compressor Rotors operate at high temperature in a closed helium environment and need to go beyond the two flexible critical speeds to achieve operating speed, 20000rpm. As traditional bearings cannot meet the engineering requirements, active magnetic bearings (AMBs) are used to support this kind of large high-speed rotor, because of its advantages such as no friction, no lubrication and sealing, long life, high attainable rotating speed and controllability [5-6].

The most significant feature of AMB is the dynamic force parameters, equivalent stiffness and damping, and stability of system are closely related to its control law. Moreover, the dynamic characteristics vary with rotating speed, even if the type of controller is changeless and the control parameters are constant [5-6]. This feature is attractive and the control parameter can be designed in advance or changed easily during rotation, such that the rotor dynamic characteristics can be actively controlled through the bearings. For example, Zhang et al. [7] isolated unbalance vibration by switching different control parameters of AMB. Hui et al. [2] design an adaptive unbalance vibration controller integrating feedforward algorithm, which can not only achieve “displacement nulling” control to effectively reduce the rotor vibration amplitude within the system bandwidth, but also realize “current nulling” control to eliminate the currents in the electromagnet windings to significantly attenuate the mechanical vibration.

However, active controllability brings difficulty to rotor dynamic analysis, because the dynamic characteristics of system is changing with rotation speed. With flexible rotor, even when the controller is known and has the simple structure, such as a PID controller, extracting the closed-loop AMB support parameters from the closed-loop transfer function is not an easy task [8]. When it comes to and complex control law, such as
sliding mode control [9], fuzzy control [10], neural network control [11] and adaptive control [12], the dynamic analysis of flexible rotor will become more difficult. Since the critical speed and unbalance response of rotor are very important for design of AMB-flexible rotor system, finding a simple way to deal with different control laws is necessary. Li [3] adopts the method of sweeping frequency of sine signal slowly produced from AMBs to excite the rotor and collect the system response signal in real time. The rotor's characteristic frequency is obtained according to the amplitude-frequency response curve. Bai et al. [13] use the electromagnetic force measurement method and the free vibration method respectively to identify the current coefficient and the displacement coefficient of the magnetic bearing, and use the identified parameters to numerical simulation. For magnetic bearing- rigid rotor system, Shen et al. [14] use multi-frequency current excitation method and least square method (LSM) to identify the structural parameters of the system. Zhou et al. [8, 15-16] use the unbalance response method to identify the stiffness and damping of the magnetic bearing under different rotation speeds. Although the above method can get accurate identification results, it is based on experiments performed by simple controllers such as PID controller. Sun et al. [17] propose a method based on the theoretical system model and the measured frequency–response model, which can estimate the unknown parameters and establish the transfer function matrix model of the AMB system. However, the algorithm is complicated.

To analyze dynamic characteristics of AMB-flexible rotor with LQR controller, a real-time integrated control system including dynamic models of flexible rotor and LQR controller at each degree of freedom of AMBs is established. The transient response in time domain and steady-state response of each node of flexible rotor under the unbalanced exciting force are calculated numerically based on this system. Then the critical speed of forward whirl is obtained which verifies the effectiveness of this method.

NOMENCLATURE

- $a$: diameter of rotor
- $\mathbf{A}, \mathbf{B}$: matrix in state-space equation based on error
- $AMB$: active magnetic bearing
- $C$: damping matrix
- $e$: vector of error
- $F_i$: electromagnetic force
- $G_1$: gain of displacement sensor
- $G_2$: gain of power amplifier
- $i$: control current
- $J$: gyroscopic matrix
- $J_p$: polar moment of inertia
- $J_d$: diameter moment of inertia
- $K$: stiffness matrix
- $k_e$: force - displacement coefficient
- $L$: length of rotor
- $LQR$: linear quadratic regulation
- $m$: total mass of rotor
- $M$: mass matrix
- $\mathbf{Q}$: generalized force vector
- $\dot{\mathbf{Q}}$: positive semidefinite matrix
- $r$: vector of reference position input
- $\hat{\mathbf{R}}$: positive definite constant matrix
- $T_i$: transformation matrix
- $\mathbf{u}$: generalized coordinates vector of displacement
- $u$: input of state-space equation
- $\mathbf{X}$: generalized coordinates of displacement
- $Y$: output of state-space equation
- $\Omega$: rotation speed
- $\Delta$: unbalanced amount
- $\varphi$: initial phase
- $\theta$: angle between the electromagnetic force and the magnetic pole
- $\alpha, \beta$: Rayleigh damping coefficient
- $\Theta$: performance index

MATHEMATICAL MODEL
Model of flexible rotor

Fig 1(a) shows a three-dimensional graphic of flexible rotor from experimental bench. $F_1, F_2$ mark the location of AMBs and $S_1, S_2$ mark the location of displacements sensors. The rotor is rotationally symmetrical and is fitted with some components including lamination, thrust discs, discs and winding. Laminated silicon steel sheets are mounted for radical AMBs and eddy current displacement sensors. Thrust discs is equipped for thrust AMB to apply force and rotor winding is mounted for driven motor. Table. 1 shows the geometrical and mass information of this rotor.

(a) Three-dimensional graphics of flexible rotor

(b) Node location of rotor

Figure 1. The geometry and elements division of rotor
### Table 1. Information of Flexible Rotor

<table>
<thead>
<tr>
<th>Items</th>
<th>Values</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L (m))</td>
<td>1.0585</td>
<td>Total length of rotor</td>
</tr>
<tr>
<td>(A (m))</td>
<td>0.024-0.036</td>
<td>Diameter of rotor</td>
</tr>
<tr>
<td>(m (kg))</td>
<td>15.8</td>
<td>Total mass of rotor</td>
</tr>
<tr>
<td>(J_p (kg \cdot m^2))</td>
<td>0.0134</td>
<td>Polar moment of inertia</td>
</tr>
<tr>
<td>(J_d (kg \cdot m^2))</td>
<td>1.04</td>
<td>Diameter moment of inertia</td>
</tr>
</tbody>
</table>

Euler-Bernoulli beam theory is employed to build the mathematical model of rotor in this article. The gyroscopic moments and rotatory inertia are taken into consideration but without considering the shear deformation. Because shear deformation of slender beam can be ignored [18].

The rotor is divided into 20 elements. Euler-Bernoulli beam element matrices are adopted to model the rotor according to the geometrical and mass information shown in Table 1. The assembled parts such as lamination stacks and discs are modeled as lumped mass onto the corresponding nodes. Ignoring translation and rotation in axial direction, each node contains two translational and two rotational DOFs. Thus the whole rotor possesses 42 DOFs. The governing equations can be obtained by Lagrange’s equation. After assembling the governing equations for all the elements and incorporating the boundary conditions, the equations of motion of rotor can be expressed as [19-20]

\[
M \ddot{u} + (C + \Omega J) \dot{u} + Ku = \begin{bmatrix} Q \end{bmatrix} (1)
\]

Where,

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, K = \begin{bmatrix} K_x & K_y \\ K_y & K_y \end{bmatrix}, Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\]

\[
C + \Omega J = \begin{bmatrix} C_x & \Omega J_x \\ -\Omega J_y & C_y \end{bmatrix}
\]

\(M, C, K\) represent square symmetric mass, damping and stiffness matrices respectively; \(C_x, K_x\) and \(C_y, K_y\) are damping and stiffness matrices in \(x, y\) directions respectively; \(J\) is gyroscopic matrix caused by gyroscopic moments; \(\Omega\) is rotation speed; \(u\) is generalized coordinate and \(u_1, u_2\) represent coordinates vector in \(x, y\) directions; \(Q\) is generalized force vector and \(Q_1, Q_2\) represent force vector in \(x, y\) directions.

In order to solve the Eq. (1) using the Runge-Kutta method. The Eq. (1) is transformed into Eq. (2) with first order form by the following transformation.

\[
\begin{bmatrix} \dot{u} \\ u \end{bmatrix}, A = \begin{bmatrix} M & K \\ K & 0 \end{bmatrix}, B = \begin{bmatrix} C + \Omega J & K \\ -K^T & 0 \end{bmatrix}, R = \begin{bmatrix} Q \\ 0 \end{bmatrix}
\]

\[
A \dot{U} + BU = \begin{bmatrix} R(t) \end{bmatrix} (2)
\]

**Active Magnetic Bearing (AMB)**

Radial AMB restrains two degrees of freedom of the rotor. Each magnetic bearing has 8 poles or 12 poles depending on the size of the load. The principle of single - degree - of - freedom AMB with differential excitation is shown in Fig. 2. The expression of electromagnetic force \(F_e\) near its balance point \(x_0\) can be obtained by linearization, as shown below [21]:

\[
F_e = F_x - F_y = k \left( \frac{(i_0 + i)^2}{(x_0 - x)^2} - \frac{(i_0 - i)^2}{(x_0 + x)^2} \right) \cos \theta = k_x i^2 + k_y x (3)
\]

Where,

\[
k_x = \frac{B \mu_0 A_i}{x_0}, \quad k_y = \frac{B \mu_0 A_0^2}{x_0^3} \cos \theta
\]

**Figure 2. The principle of single - degree - of - freedom AMB with differential excitation**

The equations of motion of rotor suspended by AMB can be obtained, when apply \(F_e\) in Eq. (3) into \(Q\) at corresponding nodes in Eq. (1). Then we need to design proper controller to control electromagnetic force, as AMB is open-loop unstable system [22].

**Controller of Linear Quadratic Regulation (LQR)**

Dynamic equation of single degree of freedom (DOF) AMB with differential excitation can be obtained as [5]

\[
\begin{bmatrix} \dot{x} \\ \dot{a}_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ a_{21} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} i 
\]

\[
Y = (1 \ 0) X (4)
\]

Where,

\[
x = [x \ x^T], \quad a_{21} = \frac{k_x}{m}, \quad b_2 = \frac{k_y}{m}
\]

Eq. (4) can be transformed as Eq. (6), if error and its derivative, as Eq. (5), is applied as generalized coordinates.
\[ Z = [z_1, z_2]^T = [e, \dot{e}]^T \]

(5)

\[ \dot{Z} = \dot{A}Z + \dot{B}u \]

(6)

Where,

\[
\hat{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -b_2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}, \quad \dot{u} = \frac{\dot{r} - a_2 \dot{r} - b_2 u}{b_2}
\]

System performance indicators function is changed as Eq. (7). Where, \( \hat{Q} \) is positive semidefinite matrix, \( \hat{R} \) is positive definite constant matrix, \( \hat{Q}, \hat{R} \) are user-defined.

\[
\Theta = \frac{1}{2} \int_0^t \left( Z^T \hat{Q} Z + u^T \hat{R} u \right) dt
\]

(7)

There is always the only optimal control input \( \hat{u} \), shown in Eq. (8), for controllable linear stationary system [23-24].

\[
\hat{u} = -\hat{R}^{-1} \hat{B} \hat{P} Z = \hat{k} Z
\]

(8)

Where,

\[
\hat{k} = [k_1, k_2]
\]

And positive definite constant matrix \( \hat{P} \) can be obtained by Riccati equation shown as Eq. (9).

\[
-\hat{P} \hat{A}^T - \hat{A}^T \hat{P} + \hat{P} \hat{B} \hat{R}^{-1} \hat{B}^T \hat{P} - \hat{Q} = 0
\]

(9)

Finally, optimal input \( i \) can be get as

\[
i = \frac{\dot{r} - a_2 \dot{r}}{b_2} + [k_1, k_2] \begin{bmatrix} e \\ \dot{e} \end{bmatrix}
\]

(10)

**METHOD AND SIMULATION**

**Rotor dynamic analysis with LQR**

The control flow chart of flexible rotor suspended by AMBs with LQR controller is shown in Fig. 3. State space equation in blue region is derived from the Eq. (2). \( G_1 \) represents the gain of displacement sensor and \( G_2 \) represents the gain of power amplifier; \( T_1 \) is transformation matrix to output the displacement in the corresponding node of support point.

**Figure 3. The control flow chart of flexible rotor suspended by AMBs with LQR controller**

Only rotor lateral vibration is considered. Therefore, at least four electromagnetic forces should be controlled as every support point needs at least two DOFs constraints from AMBs. Every electromagnetic force need a single LQR controller to constrain the displacement in the corresponding node and direction. Thus Eq. (10) can be applied to every single LQR controller shown as Fig. 4.

![LQR Controller in single DOF](image)

**Figure 4. The control system of LQR controller in single DOF**

The whole LQR controller contains four LQR controller in single DOF, shown above, at each DOF. System identification can be used to obtain critical speeds based on the system shown in Fig. 3. At different rotational speeds, the transient response in time domain of each node of the flexible rotor can be obtained by inputting unbalanced exciting force. Then steady-state response of any node at different speeds can be drawn. The rotation speed at peak point of curve is the critical speed.

**Simulation results and discussion**

In order to verify the effectiveness of the method, existing flexible rotor from experimental bench are analyzed. Conditions of simulation are shown as Table. 2. Assuming that LQR controllers’ and AMBs’ parameters are the same in four DOFs. Node 8, equipped one disc, is attached unbalance quality. Meanwhile, Nodes 3 and 19 are electromagnetic force support points.

<table>
<thead>
<tr>
<th>Items</th>
<th>Values</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta ) (kg \cdot m)</td>
<td>0.0001</td>
<td>Unbalanced amount</td>
</tr>
<tr>
<td>( \phi ) (rad)</td>
<td>( \pi / 3 )</td>
<td>Initial phase</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( 10^{-3} )</td>
<td>Rayleigh damping coefficient</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( 10^{-3} )</td>
<td>Rayleigh damping coefficient</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>25000</td>
<td>Gain of sensor</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>0.6</td>
<td>Gain of power amplifier</td>
</tr>
<tr>
<td>( k_1 )</td>
<td>( 1.78 \times 10^3 )</td>
<td>Force-displacement coefficient</td>
</tr>
<tr>
<td>( k_2 )</td>
<td>( 1.78 \times 10^2 )</td>
<td>Force-current coefficient</td>
</tr>
</tbody>
</table>

The unbalance response of flexible rotor at 0-4000 rad/s with LQR controller is plotted in Fig. 5. The steady-state response of each node on the rotor at different rotation speeds is plotted. The physical meaning of amplitude is distance between the rotor’s axis and its initial position at steady state. Backward whirl is not considered in this article. Obviously, five critical
speeds are identified, listed in Table 3. First two are rigid modal and the next three are flexible modal.

### Table 3. Identified critical speed

<table>
<thead>
<tr>
<th>Type</th>
<th>Rigid</th>
<th>Flexible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order 1</td>
<td>16.87</td>
<td>115.2</td>
</tr>
<tr>
<td>Order 2</td>
<td>28.49</td>
<td>272.2</td>
</tr>
<tr>
<td>Order 3</td>
<td>508.8</td>
<td></td>
</tr>
<tr>
<td>Order 4</td>
<td>508.8</td>
<td></td>
</tr>
</tbody>
</table>

The transient response in time domain of each node of the flexible rotor is also obtained by numerical calculation. Optionally, node 3, one of the AMB support points, and node 10, attached with one disc, at 115.2Hz and 272.2Hz are plotted in Fig. 6 and Fig. 7 respectively.

In Fig.6, the diagrams in first row show that the forward whirl radius of rotor’s axial motion changes with time. Node 3 tends to steady state after shaking for about 1.2 seconds. On the contrary, the time needed of node 10 to reach steady state is longer and the oscillation is larger because of bearing damping. The diagrams in second row show the locus of rotor’s axis. The initial position of both nodes are origin. After continuous round change, the locus of both nodes tend to be circle, because of the isotropic. Although node 10 is nearby the peak of first flexible modal, the radius of steady state circle in node 10 is \(5.45 \times 10^{-5}\) m, which is smaller than node 3, \(9.28 \times 10^{-5}\) m.

The form of Fig.7 is the similar to Fig.6. As we can see, node 3 tends to steady state after shaking for about 1.3 seconds, but the time needed of node 10 to reach steady state is longer because of bearing damping. From the locus of rotor’s axis, the locus of node 3 tend to be a circle after continuous round change. The radius of steady state circle in node 10 is \(3.11 \times 10^{-5}\) m, which is smaller than node 3, \(1.751 \times 10^{-4}\) m, because the position of node 10 is nearby the vibration node of second flexible modal.

Above analysis is consistent with the steady state plotted in Fig. 5.

### CONCLUSION

In this article, a simple and effective method based on numerical calculation of unbalanced response of a flexible rotor suspended by AMBs under the control of LQR controller is proposed to obtain the dynamic characteristics of the rotor.

As an example, the motion equation of flexible rotor with four degree of freedom is established based upon Euler-Bernoulli beam theory. LQR controller for single DOF based on measured error at support point is derived. The real-time system that
motion equation of flexible rotor controlled by LQR controller is established. Then, under the unbalanced exciting force, the steady-state response and transient response in time domain of each node of flexible rotor at 0-4000 rad/s are calculated numerically. The critical speeds are obtained from the peaks of steady-state response. Moreover, the rotor axis’ loci are plotted and compared. The simulation verifies the effectiveness of this method.

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